SNAP Centre Workshop

Graphical Vector Operations

Drawing Vectors and Vector Notation

Vectors are quantities with which a direction and magnitude are associated. **Scalar** quantities only have a magnitude associated with them.

Geometrically, a vector is represented by an arrow. For notation purposes, it is common practice to use a character in boldface or a character with a small arrow above it ($\boldsymbol{A}$ or $\vec{A}$) to denote a vector. A vector's magnitude – or its scalar value – is denoted by the same character not in boldface or without an arrow (A), or by using absolute value signs ($|\boldsymbol{A}|$ or $|\vec{A}|$), and is always positive.

Visually, vector $\boldsymbol{A}$'s magnitude is conveyed by the arrow's length. The vector's direction is conveyed by the arrow's orientation (including which end is its tip and which end is its tail). $\boldsymbol{A}$'s direction is given in terms of an angle θ above the horizontal.

Vector $\boldsymbol{A}$

θ

The two vectors pictured below have the same direction, but differ in magnitude, making them **parallel vectors**. Parallel vectors can be found by multiplying a given vector by any scalar quantity **with the exception of 0**. For example, $\boldsymbol{A}$'s magnitude is twice that of vector $\boldsymbol{B}$, or $A = 2B$. Given this, as well as the vectors' shared direction, we can say that $\boldsymbol{A} = 2\boldsymbol{B}$.

$\boldsymbol{A}$

θ

B

θ
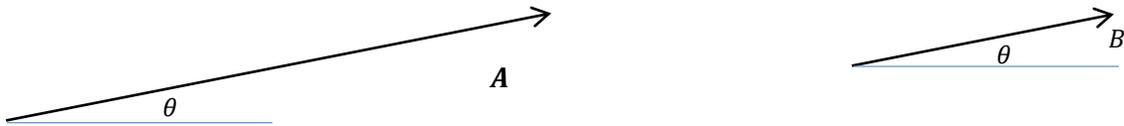
In the next example, the magnitudes of both vectors are equal ($A = C$), and their orientations are similar, but the tips and tails of the vectors are reversed, meaning the vectors differ in direction. When vectors are equal in magnitude, but opposite in direction, they're called **opposite vectors**, and are simply a given vector multiplied by the scalar quantity -1. In this particular case, $\boldsymbol{A} = -\boldsymbol{C}$.

$\boldsymbol{A}$

θ

θ

$\boldsymbol{C}$
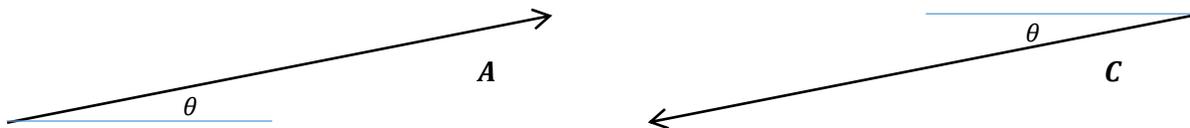
Vectors **A** and **D** are equal in magnitude ($A = D$), but differ completely in direction. There is no scalar by which **A** can be multiplied to equal **D**.

Vector placement does not have any bearing on vector equivalency. Despite vectors **B** and **E** being placed on different corners of the box pictured below (as they might be in a kinetics problem), they are equal in both magnitude and direction, making them equivalent vectors.
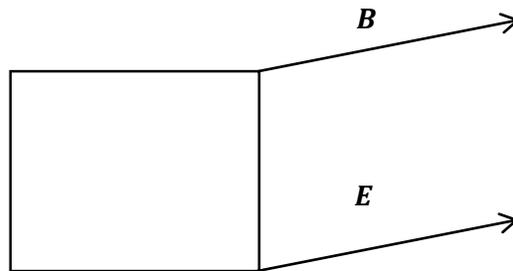
When working within a coordinate system, we are allowed to orient the coordinate axis to suit our purposes. This can make certain vector operations easier. For instance, rotating the axis pictured below allows us to treat a two dimensional vector in the x-y plane as a one dimensional vector along the x'-axis.
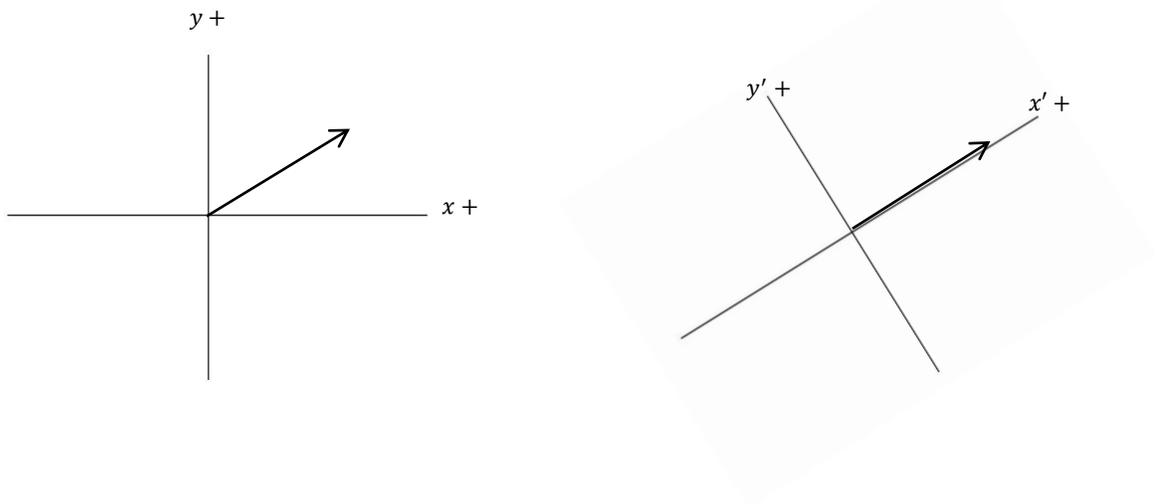
Vector Operations in One Dimension

Within one dimension, there are only two possible directions a vector can have, greatly simplifying vector operations. This single dimension can be represented visually as a number line.

$$x- \qquad \qquad \qquad \qquad x+$$
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
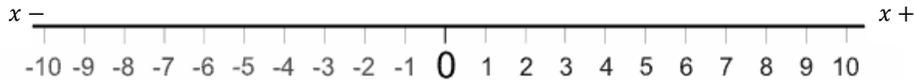
Note: Although this particular horizontal number line increases from left to right, it would be just as valid for it to be increasing from right to left, or be oriented vertically or diagonally, depending on what is most appropriate for the problem at hand.

To add vectors, we use the **tip-to-tail method**. This means we position our vector arrows so that the tip of one touches the tail of the next. After all vectors have been positioned, we draw a new, **resultant vector** by connecting the tail of our first vector to the tip of our last vector. This resultant vector represents the solution to our equation.

Example 1   *Calculate the vector sum $A + B$.*
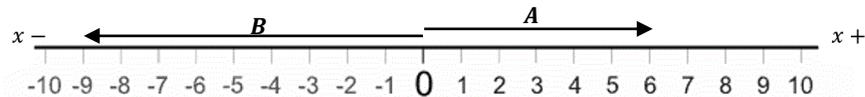
Visually, we can determine the magnitude direction of the vector arrows in the problem.

$A = +6, B = -9.$

We find the vector sum by placing A's tail at the origin of the number line, then placing B's tail at the tip of A. Our resultant vector, R, is then drawn from the tail of A to the tip of B.
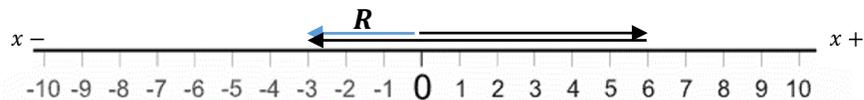
Our resultant vector R has a magnitude of 3 in the negative direction, so we can say $R = -3$.

Notice that if we had calculated the scalar sum of the vectors, $A + B$, our result would have been 15 since we would not have taken the vectors' direction into account.

We can perform the vector summation in any order and our resultant vector will not change. This allows us to make the general statement that $A + B = B + A.$

Setting Up Vector Operations in Two Dimensions

Operations involving vectors in two dimensions follow the same rules as those in one dimension. If we are given vectors A and B and add them together, we can draw the resultant vector R by placing our arrows tip-to-tail.

A   **B**

Maintaining each vector's direction, position B with its tail at the tip of A.
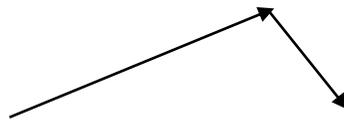
Connect the tail of A to the tip of B to draw the resultant vector R.

R
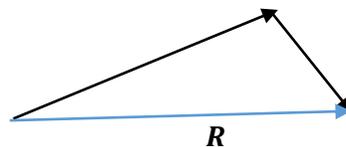
As with vector operations in one dimension, the order of addition does not change the resultant vector. A can be positioned with its tail at the tip of B, with R connecting B's tail to A's tip.

R
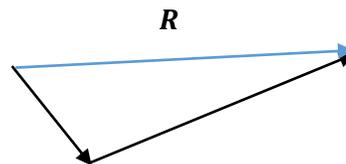
We can use opposite vectors and vector addition to find the resultant vector $B - A = S$.

The vector difference $B - A$ can be rewritten as the vector sum $B + (-A)$.

$-A$ is a vector with the same magnitude as A, but with the exact opposite direction. Using this opposite vector, and our original vector B, we use the tip-to-tail method to draw S.

$-A$ B

S
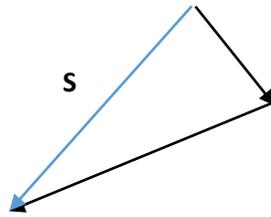
By introducing vector **C,** we can examine what happens graphically in two dimensions when more than two vectors are added or subtracted, and when vectors are multiplied.

A   **B**   **C**
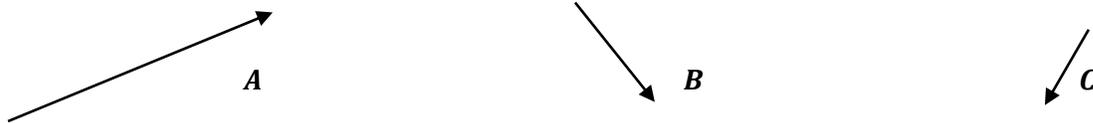
Next, we will find the resultant vector $\frac{A}{2} + B - C = P$.

Again, we place our vectors tip-to-tail, starting with $\frac{A}{2}$, which is half as long as A, and finishing with $-C$ which has the same magnitude as C, but is pointed in the opposite direction.
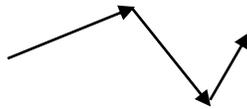
We finish by connecting the tail of our first vector to the tip of our final vector, giving us our resultant vector P.

P
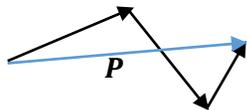
Again, we can perform the sum in any order we want. As long as the tip-to-tail method is followed, the same resultant vector P will be produced, graphically.

P
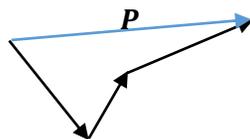
Evaluating Vector Operations Using Trigonometry

Notice that when **two** vectors are added in two dimensions, they create a triangle. If we are provided with enough information, trigonometry can be used to solve vector equations.

Example 2 $Find\ the\ vector\ \textbf{R}\ if\ \textbf{R} = \textbf{C} + \textbf{D}.$

C has a magnitude of 7, pointed in the positive x direction. D has a magnitude of 3, pointed in the negative y direction. We add them using the tip-to-tail method, with an x-y coordinate system acting as our frame of reference.
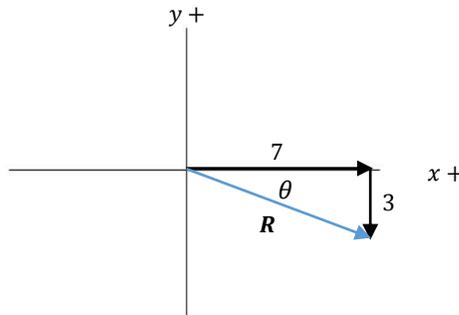
We can solve for the magnitude of R using the Pythagorean Theorem.

$$R = \sqrt{7^2 + 3^2} = \sqrt{49 + 9} = \sqrt{58}$$

Since we are looking for a vector quantity, we need to solve for R's direction as well. The inverse tangent function will allow us to solve for the angle below the positive x-axis, which is labelled as θ.

$$\theta = \tan^1 \frac{3}{7} \approx 23°$$

So, our resultant vector R has a magnitude of $\sqrt{58}$, pointed $23°$ below the positive x-axis.

Other trigonometric methods can be used to solve vector equations that don't create a right triangle.

Example 3 $Find\ the\ vector\ difference,\ \textbf{P} - \textbf{Q} = \textbf{R}.$ $|P| = 60, |Q| = 47$
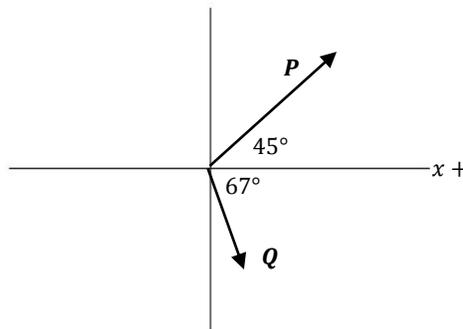
We want to convert the difference to a sum, so we position the opposite vector $-\boldsymbol{Q}$ at the tip of our $\boldsymbol{P}$ vector.
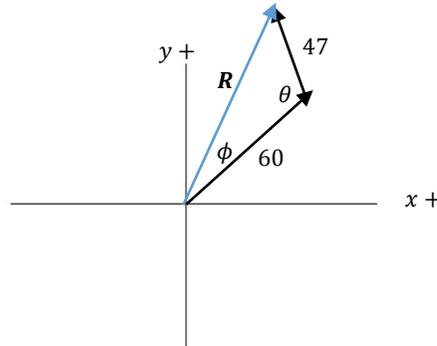
Recognizing that θ will be the sum of our angles $\theta = (67 + 45)° = 112°$, we can use the Law of Cosines to solve for R.

$$R = \sqrt{60^2 + 47^2 - 2(60)(47)\cos(112°)} \approx 89$$

Next, we can use the Law of Sines to solve for ϕ.

$$\sin\phi = \frac{Q\sin\theta}{R} = \frac{47\sin 112}{89}$$

$$\phi = \sin^{-1}\left(\frac{47\sin 112}{89}\right) \approx 29°$$

Finally, we need to add $45°$ to ϕ in order to get $\boldsymbol{R}$'s angle above the positive x-axis.

$$(29 + 45)° = 74°$$

Our resultant vector $\boldsymbol{R}$ has a magnitude of 89, pointed $74°$ above the positive x-axis.

Evaluating Vector Operations Using Rectangular Components

Although many vector problems can be solved using trigonometry alone, it becomes more difficult to do once more than two vectors are involved. By resolving vectors into their components, vector operations involving more than two vectors become much easier to solve.

Using a coordinate system as a frame of reference, we can view any given vector as the sum of two vectors, one parallel to each of our chosen axes.

Once we have resolved our vectors into their components, we can begin performing vector operations:

To perform vector addition or subtraction using rectangular components, we add or subtract all components along each axis to find the resultant component along that same axis.

To perform scalar multiplication using vector components, we multiply each component by the scalar.

Example 4 $Find\ 2\boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C} = \boldsymbol{R}\ using\ rectangular\ components.$
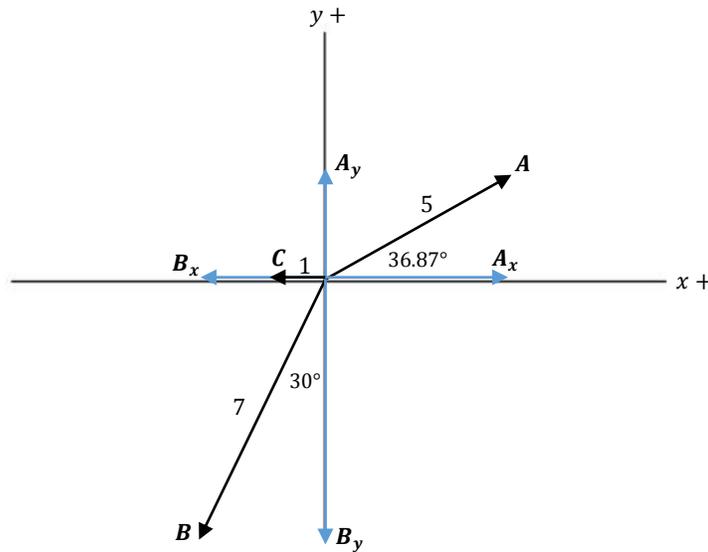
Our first step is resolving each vector into their x and y rectangular components. We do this using trigonometry.

$$A_x = 5\cos 36.87° = 4$$

We could calculate A_y, or recognize that $\boldsymbol{A_x}, \boldsymbol{A_y}$, and $\boldsymbol{A}$ form a 3-4-5 triangle.

$$A_y = 3$$

Similarly, we can see that $\boldsymbol{B_x}, \boldsymbol{B_y}$, and $\boldsymbol{B}$ form a 30-60-90 triangle, which means $B_x = \frac{B}{2}$. Also, notice that both of $\boldsymbol{B'}s$ components point in the negative direction of their respective axis. This is represented by a negative sign on each component.

$$B_x = -\frac{7}{2} = -3.5$$

$$B_y = -\sqrt{7^2 - 3.5^2} = -6.06$$

Finally, $\boldsymbol{C}$ only has one component with a magnitude of 1 in the negative x direction.

$$C_x = -1$$

With all of our components calculated, we now perform the vector operations.

$$R_x = 2A_x + B_x + C_x = 2(4) + (-3.5) + (-1)$$

$$R_x = 3.5$$

$$R_y = 2A_y + B_y + C_y = 2(3) + (-6.06) + (0)$$

$$R_y = -0.06 \approx 0$$

After rounding, we find that our resultant vector only has a positive x component of 3.5.

y +

3.5   **R**

x +
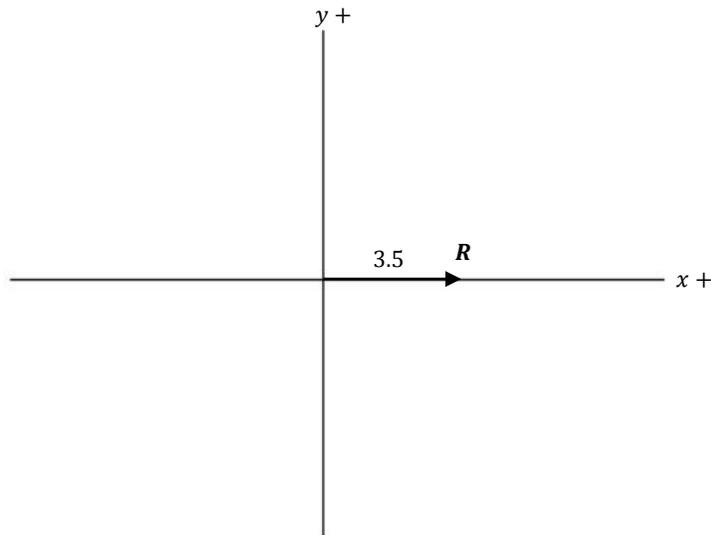
Unit Vector Notation

A **unit vector** is a vector with a magnitude of 1 in the positive direction along each axis. The unit vector for the x-axis is labelled $\hat{\imath}$ (i-hat), and the unit vector for the y-axis is labelled $\hat{\jmath}$ (j-hat).

By multiplying $\hat{\imath}$ by the x-component of a given vector, and $\hat{\jmath}$ by the y-component of a given vector, then summing the terms, we can express a given vector as the sum of their components.

Example 5   *Express $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$ from **example 4** as the sum of unit vectors.*

$$\boldsymbol{A} = 4\,\hat{\imath} + 3\hat{\jmath}$$

$$\boldsymbol{B} = -3.5\hat{\imath} - 6.06\hat{\jmath}$$

$$\boldsymbol{C} = -1\,\hat{\imath} + 0\hat{\jmath} = -1\hat{\imath}$$

Our resultant $\boldsymbol{R}$ can be calculated in terms of these unit vectors.

$$\boldsymbol{R} = (2(4) - 3.5 - 1)\,\hat{\imath} + (2(3) - 6.06)\hat{\jmath}$$

$$\boldsymbol{R} = 3.5\,\hat{\imath} + 0\hat{\jmath} = 3.5\hat{\imath}$$